On invariant distributions of circle diffeomorphisms and an equidistribution theorem for smooth potentials

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This is essentially a joint work with Michele Triestino (ENS-Lyon):


Remarks and related results by/with S. Crovisier and V. Kleptsyn will also be discussed.
Invariant distributions

• A probability measure is a point in the dual of the space of continuous functions. (Duality is realized by integration.)

• Given a manifold $M$, a $k$-distribution on $M$ is a point in the dual space of $C^k(M)$.

• A $k$-distribution $L$ is invariant under a $C^{k'}$ diffeomorphism $f : M \rightarrow M$, with $k' \geq k$, if for all $\varphi \in C^k(M)$:

$$L(\varphi) = L(\varphi \circ f).$$
An example

Assume that $x_0 \in M$ (1-dimensional) is such that

$$\sum_{n \in \mathbb{Z}} Df^n(x_0) < \infty.$$  

Then

$$\varphi \mapsto \sum_{n \in \mathbb{Z}} D\varphi(f^n(x_0)) \cdot Df^n(x_0)$$

defines an invariant 1-distribution.
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This situation arises for hyperbolic-like dynamics. Hence, it is natural to first deal with elliptic-like dynamics...
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Theorem
If $f$ is a $C^\infty$ circle diffeomorphism with irrational rotation number, then $f$ admits no invariant distribution other than (multiples of) the (unique) invariant measure.
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In dual form:

**Theorem**

For every $C^k$ function $\varphi : S^1 \to S^1$ having zero mean with respect to the $f$-invariant measure, there exists a sequence of $C^k$ functions $\psi_n : S^1 \to S^1$ such that

$$\psi_n \circ f - \psi_n \longrightarrow \varphi$$

in the $C^k$ topology.
Main Results

**Theorem**
If $f$ is a $C^{1+bv}$ circle diffeomorphism of irrational rotation number, then $f$ carries no invariant 1-distribution other than the invariant measure.
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Theorem
The theorem above is sharp in what concerns regularity of $f$. More precisely, there are $C^1$ “counterexamples” $f$ that:
- preserve an invariant Cantor set (Denjoy,...); these can be made $C^{1+\alpha}$ for all $\alpha < 1$.
- are minimal (Kodama-Matsumoto); remains unknown in class $C^{1+\alpha}$.
An Equidistribution Theorem

**Theorem**

If $f \in C^{1+bv}$ and $\varphi$ is of class $C^1$, then (for $\alpha \sim \frac{p_n}{q_n}$)

$$S_{q_n}(\varphi) - q_n \int_{S^1} \varphi d\mu \to 0.$$
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$$\frac{S_n(\varphi)}{n} \longrightarrow \int_{S^1} \varphi d\mu, \quad \varphi \text{ continuous (Weyl-Birkhoff)}$$
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$$\left| S_{q_n}(\varphi) - q_n \int_{S^1} \varphi d\mu \right| \leq \text{var}(\varphi), \quad \varphi \in C^{bv} \quad \text{(Denjoy-Koksma)}$$
Another consequence

**Theorem**
If $f$ is a $C^2$ circle diffeomorphism with irrational rotation number, then $f^{q_n}$ converges to the identity in the $C^1$ topology (M.Herman).
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**Theorem**
If $f$ is a $C^2$ circle diffeomorphism with irrational rotation number, then $f^{qn}$ converges to the identity in the $C^1$ topology (M.Herman).

**Proof.** Apply the Equidistribution Theorem to $\varphi := \log(Df) \in C^1$. Since
$$\int_{S^1} \log(Df) d\mu = 0,$$
we get
$$\log(Df^{qn}) = S_{qn}(\log(Df)) \longrightarrow 0.$$
Since $f^{qn} \rightarrow id$ in the $C^0$ topology (Denjoy), we must have $f^{qn} \rightarrow id$ in the $C^1$ topology.
Proof of the Equidistribution Theorem

• First, w.l.g., we can (and will) assume that $\varphi$ has zero mean with respect to the invariant measure.
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• Let \( \psi_m \) be such that \( \psi_m \circ f - \psi_m \rightarrow \varphi \) in the \( C^1 \) topology.

Then:

\[
S_{q_n}(\varphi) = S_{q_n}(\varphi - [\psi_m \circ f - \psi_m]) + S_{q_n}(\psi_m \circ f - \psi_m)
= S_{q_n}(\varphi - [\psi_m \circ f - \psi_m]) + \psi_m \circ f^{q_n} - \psi_m.
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= S_{q_n}(\varphi - [\psi_m \circ f - \psi_m]) + \psi_m \circ f^{q_n} - \psi_m.
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Hence,

\[
|S_{q_n}(\varphi)| \leq |S_{q_n}(\varphi - [\psi_m \circ f - \psi_m])| + |\psi_m \circ f^{q_n} - \psi_m|
\]

\[
\leq \text{var}(\varphi - [\psi_m \circ f - \psi_m]) + |\psi_m \circ f^{q_n} - \psi_m|
\]

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\leq \|\varphi - [\psi_m \circ f - \psi_m]\|_{C^1} + |\psi_m \circ f^{q_n} - \psi_m|.
\]
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\[ \psi_n := -\frac{1}{n} \sum_{i=0}^{n-1} S_i(\varphi). \]
Proof of the Main Theorem I

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$$\psi_n := -\frac{1}{n} \sum_{i=0}^{n-1} S_i(\varphi).$$

Then (remarkable identity)

$$\psi_n \circ f - \psi_n = \varphi - \frac{S_n(\varphi)}{n}$$

Hence, if $\varphi$ has zero mean, then $\psi_n$ yields the desired approximation of $\varphi$ by coboundaries in the $C^0$ topology.
Proof of the Main Theorem II

But: for generic $\varphi$, the derivative of $\psi_n \circ f - \psi_n$ explodes.
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However, for \( \varphi(x) := f(x) - x \), this works, provided \( f \in C^{1+bv} \).
Proof of the Main Theorem II

But: for generic $\varphi$, the derivative of $\psi_n \circ f - \psi_n$ explodes.

However, for $\varphi(x) := f(x) - x$, this works, provided $f \in C^{1+\text{bv}}$. Indeed, the previous identity becomes

$$\psi_n \circ f - \psi_n = \varphi - \frac{f^n(x) - x}{n}$$

and the derivative of the term $\frac{f^{q_m}(x) - x}{q_m}$ converges to zero as $m \to \infty$, because of Denjoy’s inequality.
Proof of the Main Theorem III

A very simple idea: if we wish to $C^1$-approximate $\varphi$ by functions of the form $\psi \circ f - \psi$, then $D\varphi$ should be $C^0$-approximable by functions of the form $\xi \circ f \cdot Df - \xi$. (Namely, for $\xi = D\psi$.)
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- If the functions $\xi$ had zero mean (Leb), this would solve the problem (just by integration).

- Otherwise, just subtract the integral of $\xi$...
Proof of the Main Theorem IV

End of the proof: let \( c_n := \int \xi_n \) for \( \xi_n \) such that

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Then
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(\xi_n - c_n) \circ f \cdot Df - (\xi_n - c_n) + c_n(Df - 1) \to D\varphi.
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Now recall: \( \psi_{qm} \circ f - \psi_{qm} \longrightarrow f(x) - x \) in \( C^1 \), hence

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D\psi_{qm} \circ f \cdot Df - D\psi_{qm} \longrightarrow D(f(x) - x) = Df(x) - 1
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Proof of the Main Theorem IV

End of the proof: let $c_n := \int \xi_n$ for $\xi_n$ such that

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Now recall: $\psi_{q_m} \circ f - \psi_{q_m} \to f(x) - x$ in $C^1$, hence

$$D\psi_{q_m} \circ f \cdot Df - D\psi_{q_m} \to D(f(x) - x) = Df(x) - 1.$$  

Thus,

$$(\xi_n - c_n + c_nD\psi_{q_m}) \circ f \cdot Df - (\xi_n - c_n + c_nD\psi_{q_m}) \to D\varphi.$$
Proof of the Key Fact

**Theorem**

If $\int \Phi = 0$, then $\Phi$ can be $C^0$ approximated by functions of the form $\xi \circ f \cdot Df - \xi$. 

This result is the dual statement of the fact that there are no $f$-conformal measures other than the Lebesgue measure (which is a result due to Douady, Yoccoz/Katok).
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**Theorem**
If \( \int \Phi = 0 \), then \( \Phi \) can be \( C^0 \) approximated by functions of the form \( \xi \circ f \cdot Df - \xi \).

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Definition

A measure $\nu$ is $f$-conformal if for every continuous function $\Psi$:

$$\int \Psi \circ f \cdot Df \, d\nu = \int \Psi \, d\nu.$$
Group actions

For (finitely-generated) group actions by circle diffeomorphisms without invariant measure, there should be no invariant distribution. This is well established in some cases (real-analytic, free groups), and follows (among others) from recent works with Deroin and Kleptsyn, as well as the work of Filimonov and Kleptsyn:
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- Minimal actions are ergodic with respect to the ergodic measure. (Exceptional minimal sets have zero Lebesgue measure.)
- There are no conformal measure other than Leb...
Conjugacies

The twisted cohomological equation is closely related to the next Question

Let $f$ be a $C^2$ (even $C^\infty$) circle diffeomorphism of irrational rotation number. Can $f$ be $C^2$-conjugated to diffeomorphisms arbitrarily $C^2$-close to the corresponding rotation?

In class $C^1$, this is known and easy: just use the Cohomological Identity for $\log(Df)$.
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In class $C^1$, this is known and easy: just use the Cohomological Identity for $\log(Df)$. Less trivial is that such a method works for nilpotent groups:

Theorem
Every $C^1$ action of a nilpotent group on $S^1$ (resp. $[0,1]$) is topologically conjugated to actions arbitrarily $C^1$-close to actions by rotations (resp. the trivial action).
MANY THANKS